Polar Orthotropic Inhomogeneous Circular Plates: Vibration Tailoring

Demetris Pentaras[1]
e-mail: dpentara@fau.edu

Isaac Elishakoff
e-mail: elishako@fau.edu

Department of Mechanical Engineering,
Florida Atlantic University,
Boca Raton, FL 33431-0991

Problem of matching a desired fundamental natural frequency is solved in the closed form for the polar-orthotropic inhomogeneous circular plate, which is clamped along its circumference. The vibration tailoring is performed by posing a semi-inverse eigenvalue problem. To do this, the fundamental mode shape is postulated. Namely, the analytical expression due to Lekhnitskii, and pertaining to the static deflection of the homogeneous circular plate is demanded to serve as an exact mode shape of the inhomogeneous plate. The analytical and numerical results are reported for several ratios of orthotropic coefficient. [DOI: 10.1115/1.4000410]

Keywords: vibration tailoring, circular plates, semi-inverse method, closed-form solutions

1 Introduction

Vibration tailoring has attracted investigators during the past 2 decades. The pioneering paper by Shirk et al. [1] appears to be a must for those who want to acquaint themselves with its potential. They explain the notion of aeroelastic tailoring as follows: "Aeroelastic tailoring is the embodiment of directional stiffness into an aircraft structural design to control aeroelastic deformation, static or dynamic, in such a fashion as to affect the aerodynamic and structural performance of the aircraft in a beneficial way." Authors [1] stress that the tailoring "…will no longer be regarded as an isolated phenomenon, but rather an obvious, logical extension and integral part of efficient design practice." The papers by Rehfield and Atilgan [2], Engelstad [3], Pal and Hagiwara [4], Librescu et al. [5], Constans et al. [6], and Piovan and Cortinez [7] present novel theoretical developments in implementation of this concept.

Vibration tailoring—design of a structure in such a manner so as to possess desired vibration characteristics—attracted much attention in recent years.

In this paper we deal with vibration tailoring of the clamped circular polar-orthotropic plates. It is instructive to mention several references dealing with *direct* vibration problems.

Axisymmetric vibration of clamped circular plates was studied by Prathab and Varadan [8]. According to Leissa [9], "particular attention was given to the questionable meaning of polar orthotropy at the origin ($r=0$)." Polar orthotropic circular plates possessing concentric isotropic cores were treaded by Woo et al. [10], and Rao and Ganapathi [11]. Greenberg and Stavsky [12] studied circular composite orthotropic plates. Narita [13] studied vibrations of both circular and annular plates. Vibrations of plates with variable thickness were analyzed by Laura et al. [14], Lal and Gupta [15], and Gorman [16].

Referring to tailoring, Bert [17] writes, "Although the potential for tailoring of laminates is usually mentioned as one of their advantages, most optimization to date has been on an ad hoc basis and has not considered dynamic criteria (objective functions or constraints)." Rao and Singh [18] attempted to remedy this situation in a formal optimization using nonlinear mathematical programming. They considered minimum-weight design with constraints on minimum fundamental natural frequency, minimum buckling load, and maximum static deflection." Bert [19,20] posed the problem of optimization angle that maximizes the natural frequency.

This study is somewhat resonating with those by Rao and Singh [18] and Bert [19,20]; although we do not study the optimal design of the structure under consideration, namely, the circular polar-orthotropic plates, we deal with the fundamental of natural frequency. We are concerned with the design that demands the fundamental frequency to equal the preselected value that is dictated by consideration of resonance; for example, the designer wants natural frequency to be removed from the excitation frequency by a given interval.

2 Basic Equations

Consider first a clamped-clamped orthotropic plate under statically applied uniformly distributed load of intensity q_0. The deflection is given by Lekhnitskii's [21] classic formula

$$w(r) = \frac{q_0 a^4}{8(9-k^2)(1+k)D_r}[3-k-4(r/R)^{k+1}+(1+k)(r/R)^4] \tag{1}$$

where k is the orthotropy coefficient equal to $\sqrt{D_\theta/D_r}$, D_r and D_θ being, respectively, radial and circumferential flexural rigidities, r=radial coordinate, and R=radius of the plate. In this formula, D_r and D_θ are considered to be the constants.

Consider now the free vibration of the circular *inhomogeneous* plate. The governing differential equation reads

$$\begin{aligned} &rD_r(r)W^{IV} + \left(2D_r(r) + 2r\frac{dD_r(r)}{dr} + \nu_\theta D_r(r) - \nu_r D_\theta(r)\right)W''' \\ &- \left[\frac{d}{dr}\left(-D_r(r) - r\frac{dD_r(r)}{dr} - \nu_\theta D_r(r) + \nu_r D_\theta(r)\right) - \nu_\theta\frac{dD_r(r)}{dr} \right. \\ &\left. + \frac{1}{r}D_\theta(r)\right]W'' - \left[\frac{d}{dr}\left(-\nu_\theta\frac{dD_r(r)}{dr} + \frac{1}{r}D_\theta(r)\right)\right]W' \\ &= r\rho h\omega^2 W(r) \end{aligned} \tag{2}$$

where $W(r)$ is the mode shape, $D_r(r)$ and $D_\theta(r)$ are, respectively, variable radial and circumferential flexural rigidities, depending on the radial coordinate r, but being taken as independent of the circumferential coordinate θ, ν_r, ν_θ are Poisson's ratios of ortho-

[1]Corresponding author.

Contributed by the Applied Mechanics Division of ASME for publication in the JOURNAL OF APPLIED MECHANICS. Manuscript received May 22, 2008; final manuscript received July 17, 2009; published online March 5, 2010. Assoc. Editor: Wei-Chau Xie.

tropic plate, considered to be constants, ρ is the material density, h is the thickness, and ω is the natural frequency. Our task is to design the plate in such a manner so as it to posses the prespecified fundamental frequency Ω.

To achieve the desired design of a circular plate, we demand the fundamental mode shape, $W(r)$, in the governing differential equation (2) to coincide with the expression in brackets in Eq. (1) or

$$W(r) = 3 - k - 4(r/R)^{k+1} + (1+k)(r/R)^4 \quad (3)$$

This implies that the mode shape of the inhomogeneous plate is proportional to the static displacement of the homogeneous one.

We deal exclusively with the case when the mode shape (2) is a polynomial function. Therefore, we let the value k in Eq. (3) as equal some integer m:

$$W(r) = 3 - m - 4(r/R)^{m+1} + (1+m)(r/R)^4 \quad (4)$$

and examine if there exists a non-negative-valued variations of $D_r(r)$ and $D_\theta(r)$ that taken together with mode shape (4) satisfy the governing differential equation (2).

It should be noted that the proposed method constitutes the *semi-inverse* method. This is due to the fact that the fundamental mode shape, which is ordinarily obtained as the result of the direct vibration analysis, is being assumed here to be given. One may ask, "Why demand that the free vibration mode shape be the same as the static deflection of a uniformly loaded plate?" Indeed, one may think that what we want to do is a mystery. In reality, however, our approach can be justified as follows. Homogeneous structures do not allow the vibration mode to be a polynomial function; however, inhomogeneous structures, by their virtue, contain *more parameters* than their homogeneous counterparts. Therefore, one can solve the problem of *vibration tailoring*, in which a given function, including the *static displacement* of the corresponding homogeneous structure, may serve as the *exact* mode shape. We show that this *unusual* approach works well and yields nontrivial and significant results. We hope that this method will be both widely utilized and further developed by other researchers.

3 Semi-Inverse Method of Solution Associated With $m=1$

For $m=1$, the mode shape in Eq. (4) reads

$$W(r) = 2 - 4(r/R)^2 + 2(r/R)^4 \quad (5)$$

The flexural rigidities are sought as polynomials of the fourth order

$$D_r(r) = \sum_{j=0}^{4} b_j r^j \quad (6)$$

$$D_\theta(r) = k^2 D_r(r) \quad (7)$$

where k is taken as constant. The substitution of Eqs. (6), (7), and (4) into a governing differential equation leads to the following polynomial equation:

$$S_0 + S_1 r + S_2 r^2 + S_3 r^3 + S_4 r^4 + S_5 r^5 = 0 \quad (8)$$

where

$$S_0 = -4k^2R^2b_1 + 8R^2b_1 + 8\nu_\theta R^2 b_1 - 4v_r R^2k^2b_1 \quad (9)$$

$$S_1 = 24R^2b_2 - 24\nu_\theta b_0 + 8k^2b_0 - 72b_0 - 8k^2b_0 + 24\nu_\theta R^2b_2 - 8k^2\nu_r R^2b_2 + 24k^2\nu_r b_0 - 8k^2R^2b_2 + \rho h\omega^2R^4 \quad (10)$$

$$S_2 = 12k^2b_1 + 48\nu_\theta R^2b_3 - 12k^2\nu_r R^2b_3 + 36k^2\nu_r b_1 - 144b_1 + 48R^2b_3 - 12k^2R^2b_3 - 48\nu_\theta b_1 \quad (11)$$

$$S_3 = 16k^2b_2 + 48k^2\nu_r b_2 - 16k^2R^2b_4 + 80\nu_\theta R^2b_4 + 80R^2b_4 - 80v_\theta b_2 - 16v_r k^2R^2b_4 - 240b_2 - 2\rho h\omega^2R^2 \quad (12)$$

$$S_4 = 20k^2b_3 + 60k^2\nu_r b_3 - 120\nu_\theta b_3 + 360b_3 \quad (13)$$

$$S_5 = -504b_4 - 168\nu_\theta b_4 + 24k^2b_4 - 72k^2v_r b_4 + \rho h\omega^2 \quad (14)$$

From Eq. (14) we get the relationship between the natural frequency squared ω^2 and the coefficient b_4:

$$\omega^2 = 24(21 + 7\nu_\theta - 3\nu_r k^2 - 21k^2)b_4/\rho h \quad (15)$$

Equation (13) can be factored as follows, $(20k^2+60k^2\nu_r-120\nu_\theta+360)b_3=0$. Since the expression in the parentheses does not necessarily equal zero, we conclude that in order for Eq. (13) to be valid, b_3 must vanish. The equation resulting from substitution of Eq. (15) into Eq. (12) results in the formula for b_2, as related to b_4:

$$b_2 = -2\frac{(k^2 + 4k^2\nu_r - 8\nu_\theta - 29)}{(-5\nu_\theta + k^2 + 3k^2\nu_r - 15)}R^2b_4 \quad (16)$$

Equation (11) leads to the conclusion that $b_1=0$. The equation resulting for substitution of Eqs. (15) and (16) into Eq. (10) results in the expression for b_0:

$$b_0 = -\frac{b_4}{(k^2 - 5\nu_\theta + 3k^2\nu_r - 15)(k^2 - 3\nu_\theta + 3k^2\nu_r - 9)}(-242k^2\nu_r - 44k^2 + 408\nu_\theta - 68k^2\nu_\theta\nu_r - 14k^2 + 57\nu_\theta^2 + 8k^4\nu_r + 19k^4\nu_r + k^4 + 771) \quad (17)$$

In view of the relationship $\nu_\theta=k^2\nu_r$, we get the following final expression for the flexural rigidity:

$$D_r(r) = \frac{b_4}{(2k^2\nu_r - k^2 + 15)(k^2 - 9)} \times [(-18k^2\nu_r + 2k^4\nu_r + 24k^2 - k^4 - 135)r^4 + (72k^2\nu_r + 522 + 2k^4 - 8k^4 - 8k^4\nu_r - 76k^2)R^2r^2 + (-8k^4\nu_r^2 - k^4 - 771R^4 + 44k^2 + 6k^4\nu_r - 166k^2\nu_r)R^4r^4] \quad (18)$$

Figure 1 depicts the variation in $D_r(r)$ as a function of r, for various values of k, for ν_r fixed at 0.35.

4 Semi-Inverse Method of Solution Associated With $m=2$

For $m=2$, the mode shape in Eq. (4) reads

$$W(r) = 1 - 4(r/R)^3 + 3(r/R)^4 \quad (19)$$

The result of substitution of Eq. (19) in coincidence of Eqs. (6) and (7) yields the equation:

$$T_0 + T_1 r + T_2 r^2 + T_3 r^3 + T_4 r^4 + T_5 r^5 = 0 \quad (20)$$

where

$$T_0 = 24v_\theta Rb_0 - 24k^2\nu_r Rb_0 + 48Rb_0 - 12k^2Rb_0 \quad (21)$$

$$T_1 = 72k^2\nu_r b_0 - 216b_0 + 24k^2b_0 - 24k^2Rb_1 - 48k^2R\nu_r b_1 + 72v_\theta Rb_1 - 72\nu_\theta b_0 + 144Rb_1 + \rho h\omega^2R^4 \quad (22)$$

$$T_2 = -432b_1 - 144\nu_\theta b_1 + 108\nu_r k^2b_1 + 144\nu_\theta Rb_2 - 72Rk^2v_r b_2 + 288b_2R + 36k^2b_1 - 36k^2Rb_2 \quad (23)$$

$$T_3 = 144k^2v_r b_2 - 720b_2 - 240v_\theta b_2 + 240\nu_\theta Rb_3 - 48k^2Rb_3 - 96k^2Rv_r b_3 + 480Rb_3 + 48k^2b_2 \quad (24)$$

$$T_4 = 180kv_r b_3 - 360\nu_\theta b_3 + 360R\nu_\theta b_4 - 1080b_3 + 720Rb_4 - 120k^2Rv_r b_4 + 60k^2b_3 - 60Rk^2b_4 - 4\rho h\omega^2R \quad (25)$$

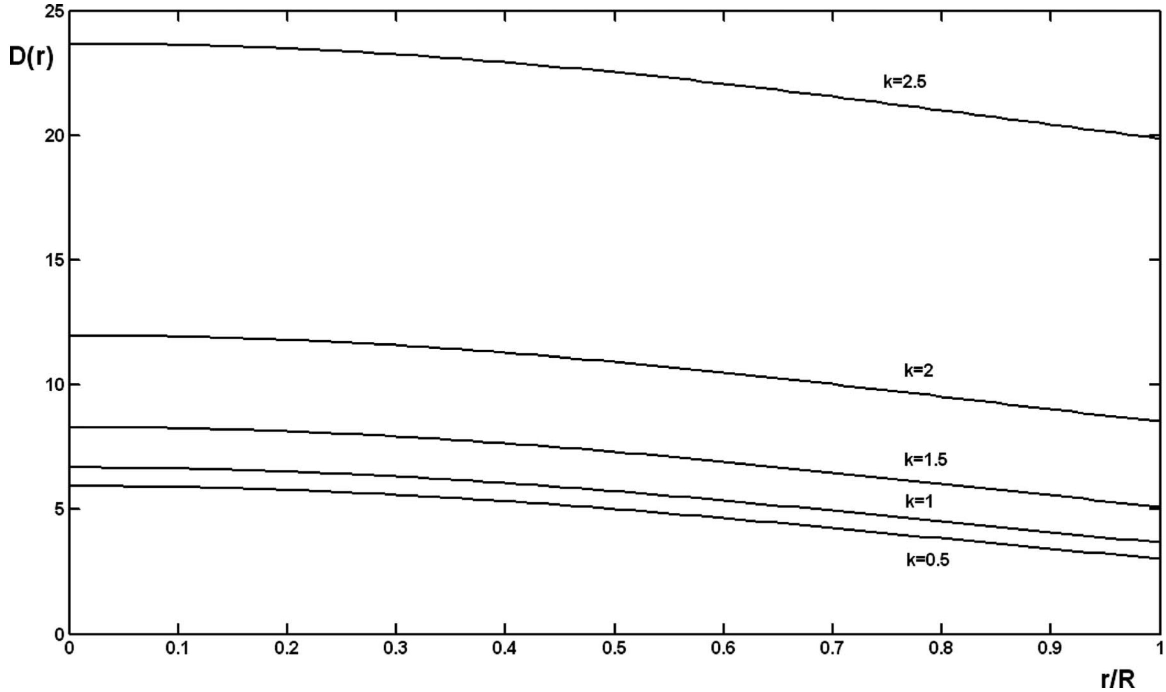

Fig. 1 Variation in $D(r)$ versus nondimensional radial coordinate r/R for various values of k and $\nu_r=0.35$

$$T_5 = 72k^2 b_4 + 216k^2 \nu_r b_4 - 504\nu_\theta b_4 - 1512 b_4 + 3\rho h\omega^2 \quad (26)$$

From Eq. (26) we get the expression for natural frequency squared ω^2:

$$\omega^2 = 24(21 + 7\nu_\theta - 3\nu_r k^2 - 21k^2)b_4/\rho h \quad (27)$$

It is significant to note, that the expression (27) coincides with Eq. (15) for the squared natural frequency. This means that although the mode shapes in Eqs. (5) and (15) are *different*, the natural frequency is obtained as the *same* analytical expression. This does not imply that the solution is immune to the mode shape. It should be anticipated that the variations of flexural rigidities will be different in these two cases. Substitution of Eq. (27) into Eq. (25) yields

$$b_3 = -\frac{1}{5}\frac{(3k^2 + 14k^2\nu_r - 26\upsilon_\theta - 21)}{(-6\nu_\theta + k^2 + 3k^2\nu_r - 18)}Rb_4 \quad (28)$$

Substituting Eq. (28) into Eq. (24) we get

$$b_2 = -\frac{b_4R^2}{5(k^2 - 6\nu_\theta + 3k^2\nu_r - 18)(k^2 - 5\nu_\theta + 3k^2\nu_r - 15)}(-26\nu_\theta + 3k^2 - 107 + 14k^2\nu_r)(-5\nu_\theta + 2k^2\upsilon_r - 10 + k^2) \quad (29)$$

Eq. (23) leads to the coefficient b_1:

$$b_1 = -\frac{b_4R^3}{5(3k^2\nu_r - 5\nu_\theta - 15 + k^2)(-6\nu_\theta + k^2 + 3k^2\nu_r - 18)(-5\nu_\theta + k^2 + 3k^2\nu_r - 15)}(3k^2 - 108 + 14k^2\nu_r - 26\nu_\theta)(-10 + k^2 - 5\nu_\theta + 2k^2\nu_r) \times(-4\nu_\theta + k^2 - 8 + 2k^2\nu_r) \quad (30)$$

From Eq. (21), we conclude that $b_0=0$. Equation (22) yields another expression for b_1:

$$b_1 = -\frac{b_4(-7\nu_\theta + k^2 + 3k^2\nu_r - 21)(6 + \nu_\theta)R^3}{-6 + 2k^2\nu_r + k^2 - 3\nu_\theta} \quad (31)$$

In order to resolve the contradiction of Eqs. (30) and (31), we demand expressions for b_1 to coincide. This requirement results in the following polynomial equation:

$$\sum_{j=0}^{4} U_j \nu_r^j = 0 \quad (32)$$

where

$$U_0 = 2k^8 - 150k^6 + 4899k^4 - 64386k^2 + 288360 \quad (33)$$

$$U_1 = -20k^8 + 1188k^6 - 24576k^4 + 152298k^2 \quad (34)$$

$$U_2 = 70k^8 - 2868k^6 + 28137k^4 \quad (35)$$

$$U_3 = -100k^8 + 2082k^6 \quad (36)$$

$$U_4 = 48k^8 \quad (37)$$

Numerical evaluation of the roots of Eq. (32) reveals the following: For $k=1$, $k=2$, or $k=3$ it does not possess real positive roots for ν_r. For $k=4$ the only positive root of Eq. (32) is $\nu_r=0.1080$. Figure 2 portrays the appropriate variation in the radial flexural rigidity $D_r(r)$.

For $k=5$, Eq. (32) yields a positive root, $\nu_r=0.4431$ (Fig. 3); $k=6$ corresponds to two positive roots $\nu_r=0.2574$ and ν_r

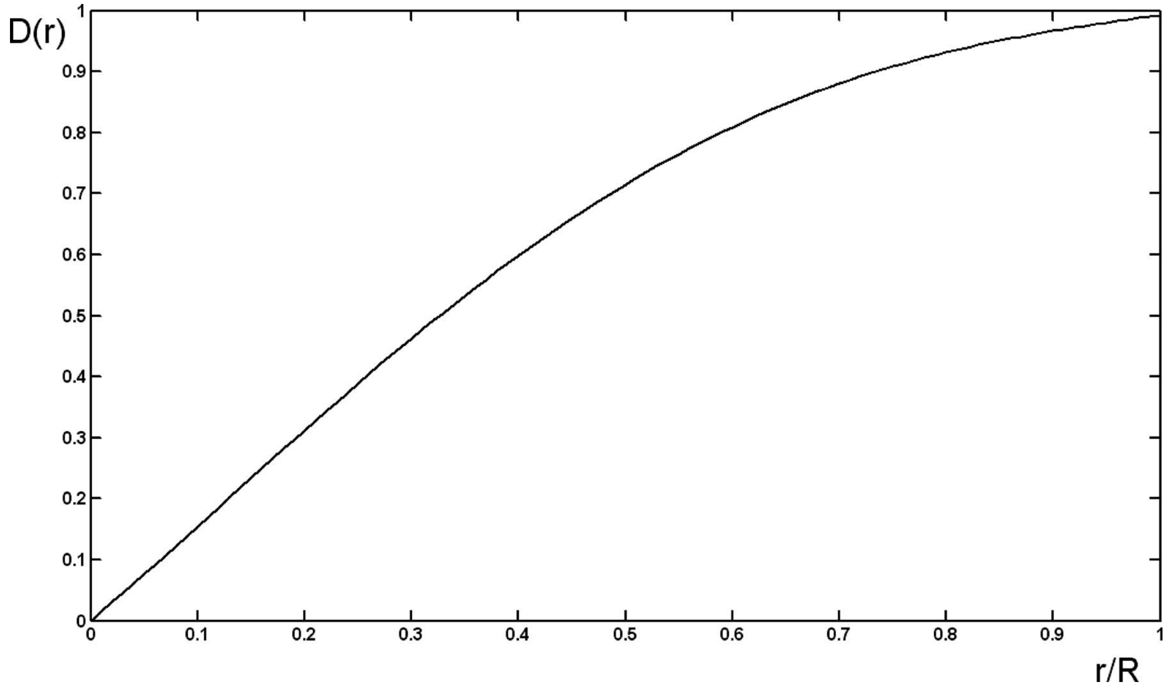

Fig. 2 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=4$ and $\nu_r=0.1080$

$=0.5628$ (Fig. 4). Likewise, Eq. (32) has two positive roots $\nu_r=0.2964$ and $\nu_r=0.6498$ for $k=7$. Similarly, two roots $\nu_r=0.3262$ and $\nu_r=0.7162$ match to the orthotropy coefficient $k=8$. In addition, the value $k=9$ is represented with roots $\nu_r=0.3512$ and $\nu_r=0.7673$, whereas $k=10$ corresponds to $\nu_r=0.3732$ and $\nu_r=0.8068$. Figures for cases $k=7$, 8, 9, and 10 are not reproduced here to save space.

5 Semi-Inverse Method of Solution Associated With $m=4$

For $m=4$, the mode shape in Eq. (4) becomes

$$W(r) = -1 + 5(r/R)^4 - 4(r/R)^5 \tag{38}$$

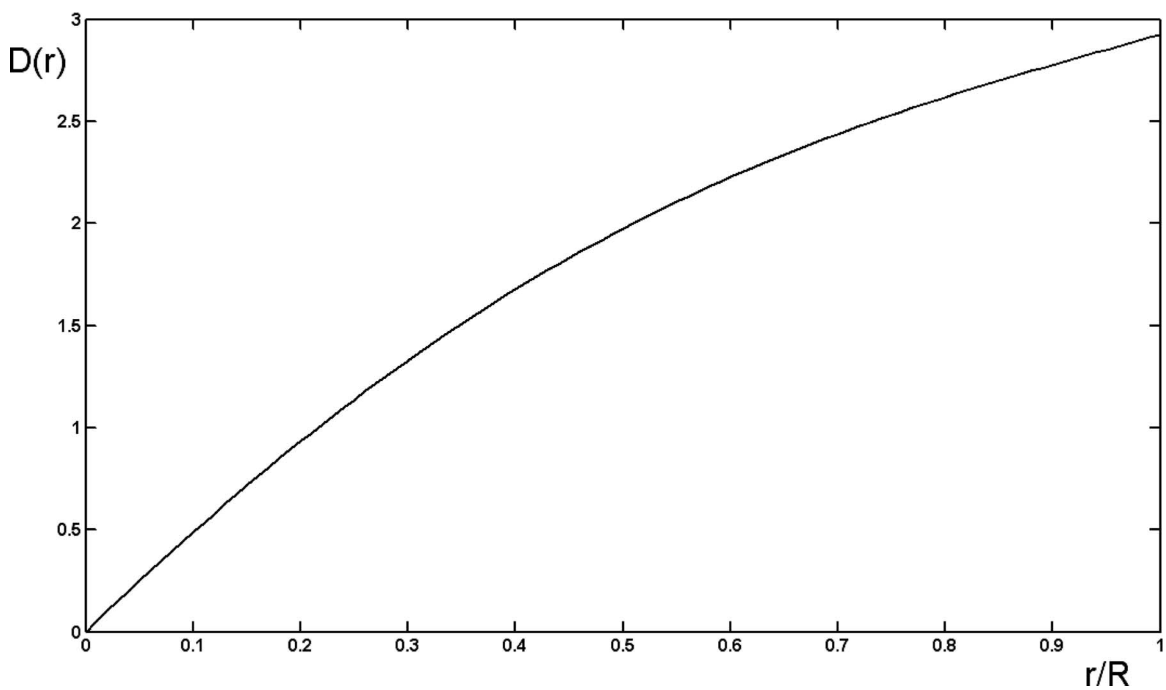

Fig. 3 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=5$ and $\nu_r=0.4431$

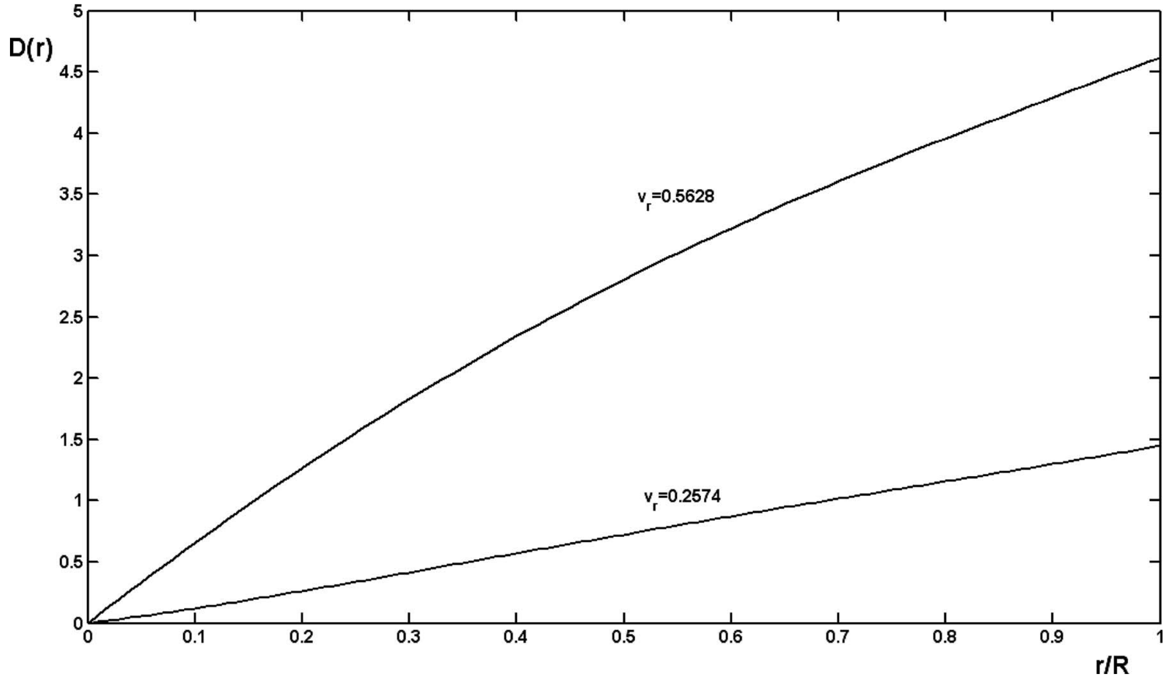

Fig. 4 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=6$

The result of substitution of Eq. (38) in coincidence of Eqs. (6) and (7) yields the equation:

$$V_0 + V_1 r + V_2 r^2 + V_3 r^3 + V_4 r^4 + V_5 r^5 + V_6 r^6 = 0 \quad (39)$$

where

$$V_0 = 0 \quad (40)$$

$$V_1 = \rho h \omega^2 R^5 + 360 b_0 R - 120 k^2 R \nu_r + 360 b_0 R + 120 b_0 \nu_\theta R - 40 k^2 b_0 R \quad (41)$$

$$V_2 = -180 k^2 b_1 \nu_r R + 720 b_1 R + 240 k^2 b_0 \nu_r - 60 k^2 b_0 \nu_r - 240 b_0 \nu_\theta - 960 b_0 + 240 R \nu_\theta b_1 + 60 k^2 b_0 \quad (42)$$

$$V_3 = -80 k^2 b_2 R - 400 b_1 \nu_\theta + 320 k^2 b_1 \nu_r - 240 k^2 b_2 \nu_r R + 400 b_2 \nu_\theta R + 1200 R b_2 - 1600 b_1 - 80 k^2 b_1 \quad (43)$$

$$V_4 = 400 k^2 b_2 \nu_r + 600 b_3 \nu_\theta R + 100 k^2 b_2 - 2400 b_2 - 600 b_2 \nu_\theta - 300 k^2 b_3 R \nu_r + 1800 b_3 R - 100 k^2 b_3 R \quad (44)$$

$$V_5 = 2520 b_4 R - 120 k^2 b_4 R - 2660 b_3 - 360 k^2 b_4 \nu_r R - 5 \rho h \omega^2 R - 3360 b_3 - 360 k^2 \nu_r R b_4 + 480 k^2 b_3 \nu_r + 840 b_4 \nu_\theta R + 120 k^2 b_3 - 840 b_3 \nu_\theta \quad (45)$$

$$V_6 = 560 k^2 b_4 \nu_r - 4480 b_4 - 1120 b_4 \nu_\theta + 140 k^2 b_4 + 4 \rho h \omega^2 \quad (46)$$

From Eq. (46) we get the expression for natural frequency squared ω^2:

$$\omega^2 = 35(32 + 8\nu_\theta - 4\nu_r k^2 - k^2) b_4 / \rho h \quad (47)$$

Substitution of Eq. (47) into Eq. (48) leads to

$$b_3 = -\frac{1}{24} \frac{(11k^2 + 68k^2\nu_r - 112\nu_\theta - 616)}{(-7\nu_\theta + k^2 + 4k^2\nu_r - 28)} R b_4 \quad (48)$$

Substituting Eq. (48) into Eq. (44) we get

$$b_2 = -\frac{1}{24} \frac{(68k^2\nu_r - 616 - 112\nu_\theta + 11k^2)(-18 - 6\nu_\theta + 3k^2\nu_r + k^2)}{(-7\nu_\theta + k^2 - 28 + 4k^2\nu_r)(4k^2\nu_r - 6\nu_\theta - 24 + k^2)} \times R^2 b_4 \quad (49)$$

Eq. (43) leads to the coefficient b_1:

$$b_1 = W/X \quad (50)$$

where

$$W = -(-5\nu_\theta + k^2 - 15 + 3k^2\nu_r)(-6\nu_\theta - 18 + k^2 + 3k^2\nu_r)(-112\nu_\theta - 616 + 11k^2 + 68k^2\nu_r) R^3 b_4 \quad (51)$$

$$X = 24(-7\nu_\theta + k^2 + 4k^2\nu_r + k^2)(-6\nu_\theta + 4k^2\nu_r + k^2 - 24)(k^2 + 4k^2\nu_r - 5\nu_\theta - 20) \quad (52)$$

From Eq. (41), we get

$$b_0 = -\frac{7}{8} \frac{(4k^2\nu_r - 32 - 8\nu_\theta + k^2)}{(-3\nu_\theta + 3k^2\nu_r - 9 + k^2)} R^4 b_4 \quad (53)$$

Eq. (42) yields another expression for b_1:

$$b_1 = Y/Z \quad (54)$$

where

$$Y = -7(8k^4\nu_r + 512 + 256\nu_\theta + k^4 - 12\nu_\theta k^2 - 48k^2 - 48k^2\nu_r\nu_\theta + 32\nu_\theta^2 - 192k^2\nu_r + 16k^4\nu_r^2) \quad (55)$$

$$K = 8(-3\nu_\theta + k^2 - 9 + 3k^2\nu_r)(3k^2\nu_r - 4\nu_\theta - 12 + k^2) \quad (56)$$

We demand the expressions for b_1 in Eqs. (50) and (54) to be equal. This results in the following polynomial equation:

$$\sum_{j=0}^{4} \Phi_j \nu_r^j = 0 \quad (57)$$

where

$$\Phi_0 = 10k^{10} - 1310k^8 + 74235k^6 - 2020770k^4 + 25946040k^2 - 126544320 \quad (58)$$

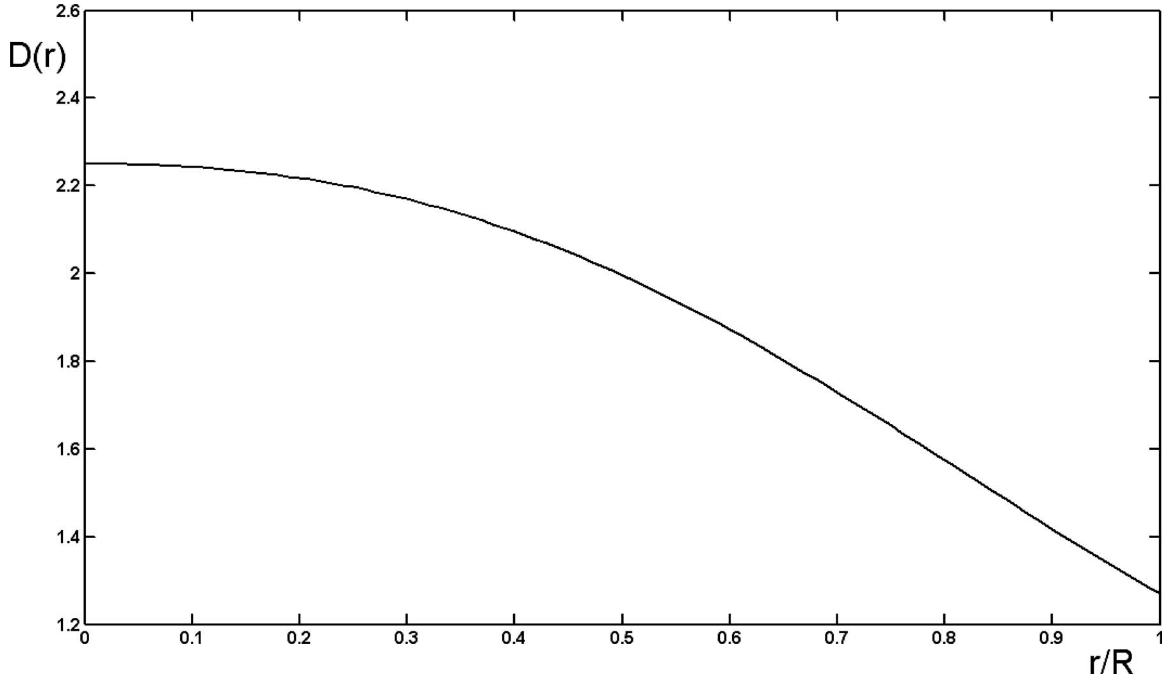

Fig. 5 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=4$ and $\nu_r=1/32$

$$\Phi_1 = -100k^{10} + 10740k^8 - 10740k^8 - 436740k^6 + 7439322k^4 - 44645136k^2 \quad (59)$$

$$\Phi_2 = 350k^{10} - 28300k^8 + 722133k^6 - 5670240k^4 \quad (60)$$

$$\Phi_3 = -500k^{10} - 26286k^8 - 303972k^6 \quad (61)$$

$$\Phi_4 = -5688k^8 + 240k^{10} \quad (62)$$

Numerical evaluation of the roots of Eq. (57) reveals the following: For $k=1$, $k=2$, or $k=3$, Eq. (57) does not possess real positive roots for ν_r. For $k=4$, the only positive roots of Eq. (57) is $\nu_r=1/32$. Figure 5 shows the variation in $D(r)$ versus nondimensional radial r/R. For $k=5$, Eq. (57) does not have real positive roots. The value $k=6$ corresponds to a positive root $\nu_r=0.3644$ (Fig. 6). Equation (57) has two positive roots $\nu_r=0.2228$ and $\nu_r=0.4839$ for $k=7$ (Fig. 7). Also, two roots $\nu_r=0.2635$ and $\nu_r=0.5799$ correspond to $k=8$. The value $k=9$ is associated with roots $\nu_r=0.2958$ and $\nu_r=0.6570$, whereas $k=10$ is associated with $\nu_r=0.3239$ and $\nu_r=0.7175$. Figures associated with cases $k=8$, 9, and 10 are not reproduced to save space.

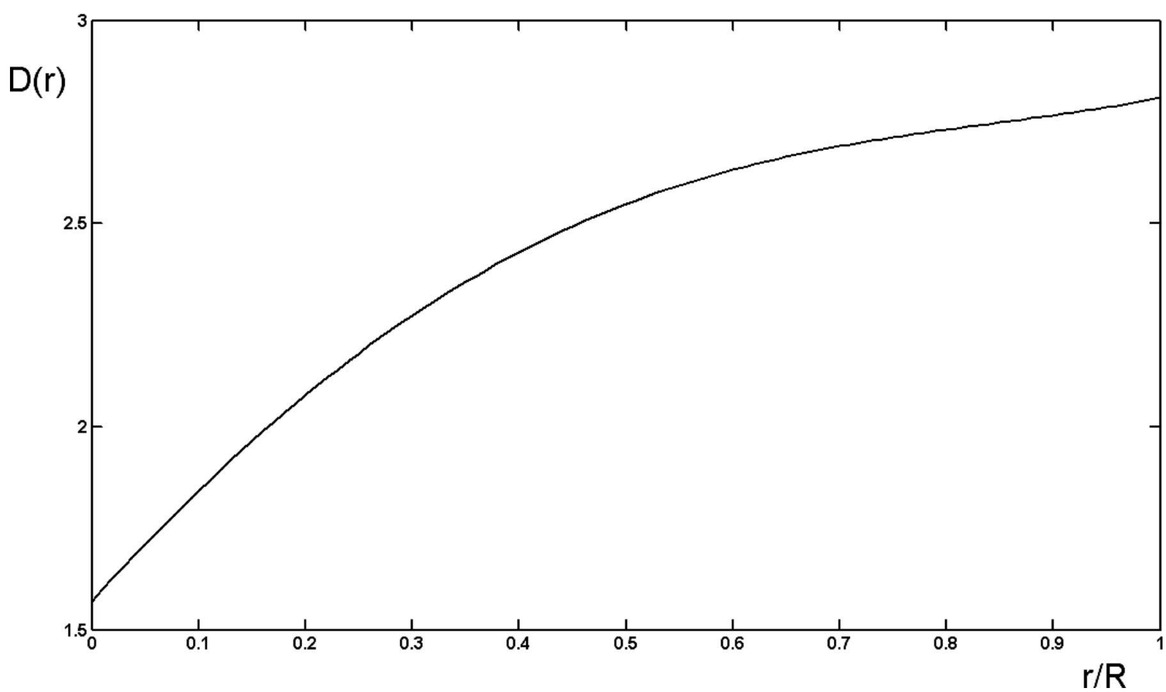

Fig. 6 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=6$ and $\nu_r=0.3644$

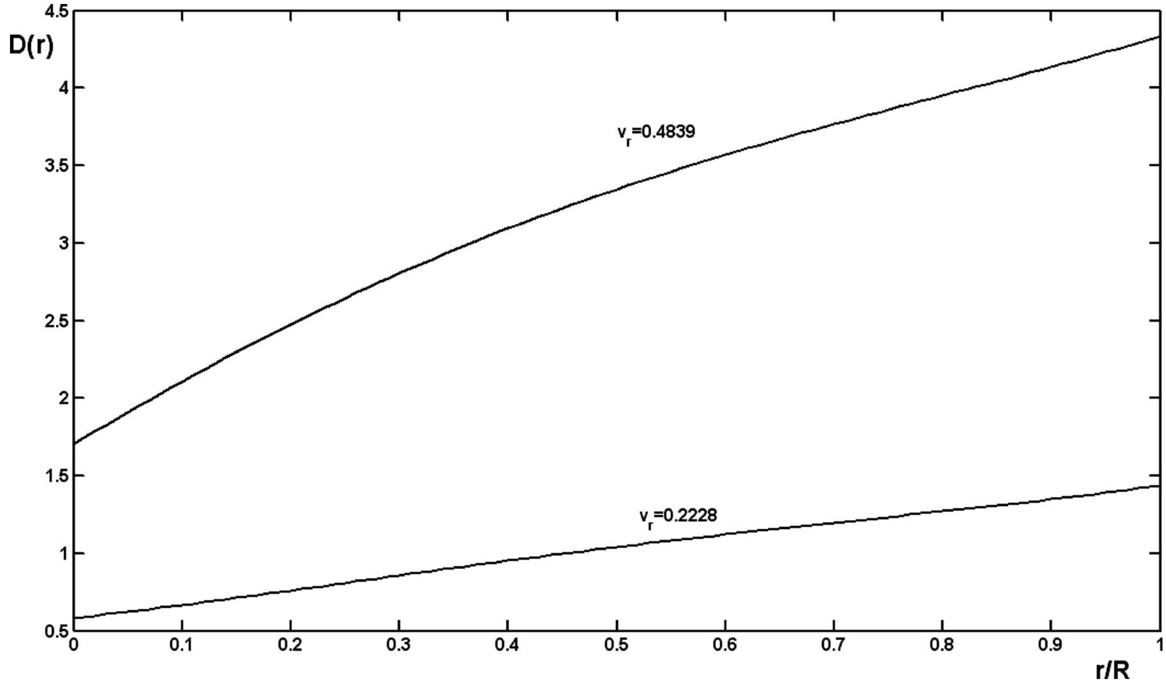

Fig. 7 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=7$

6 Semi-Inverse Method of Solution Associated With $m=5$

For $m=5$, the mode shape in Eq. (4) reads

$$W(r) = 1 - 3(r/R)^4 + 2(r/R)^6 \quad (63)$$

The result of substitution of Eq. (63) in coincidence of Eqs. (6) and (7) into the governing equation yields the following polynomial equation:

$$\Theta_0 + \Theta_1 r + \Theta_2 r^2 + \Theta_3 r^3 + \Theta_4 r^4 + \Theta_5 r^5 + \Theta_6 r^6 + \Theta_7 r^7 = 0 \quad (64)$$

where

$$\Theta_0 = 0 \quad (65)$$

$$\Theta_1 = \rho h \omega^2 R^6 - 72k^2 b_0 \nu_r R^2 - 24k^2 b_0 R^2 + 72 b_0 \nu_\theta R^2 + 216 b_0 R^2 \quad (66)$$

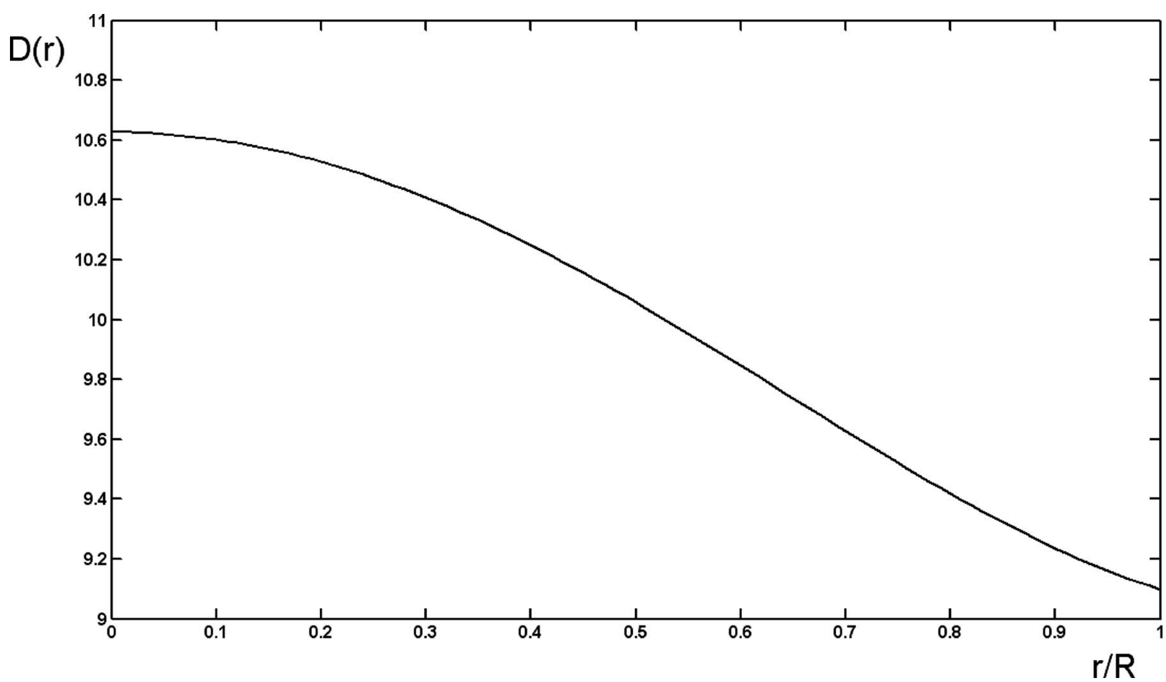

Fig. 8 Variation in $D(r)$ versus nondimensional radial coordinate r/R for $k=10$ and $\nu_r=0.9338$

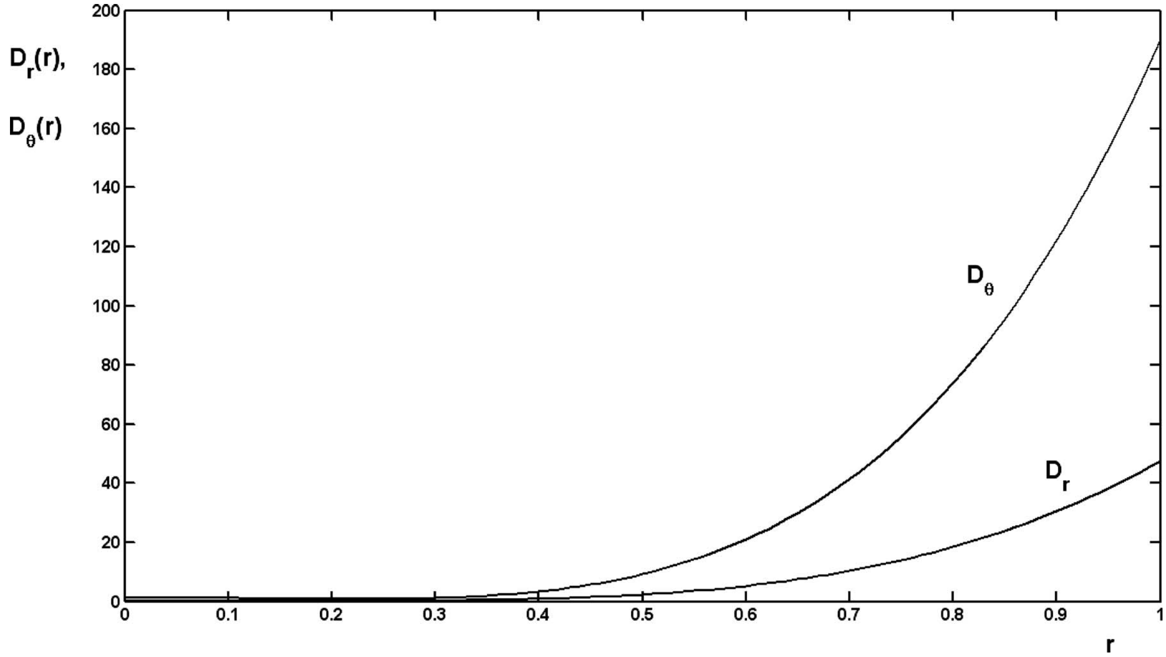

Fig. 9 Variation in $D_r(r)$ and $D_\theta(r)$ versus radial coordinate r for Eq. (82)

$$\Theta_2 = -36k^2b_1R^2 + 144b_1\nu_\theta R^2 + 432b_1R^2 - 108k^2b_1\nu_rR^2 \quad (67)$$

$$\Theta_3 = -48k^2b_2R^2 - 240b_0\nu_\theta - 144k^2b_2\nu_rR^2 + 720b_2R^2 + 48k^2b_0 - 1200b_0 + 240b_2\nu_\theta R^2 + 240k^2b_0\nu_r \quad (68)$$

$$\Theta_4 = -1800b_1 - 360b_1\nu_\theta + 60k^2b_1 + 300k^2b_1\nu_r + 360b_3\nu_\theta R^2 + 1080b_3R^2 - 180k^2b_3\nu_rR^2 - 60k^2b_3R^2 \quad (69)$$

$$\Theta_5 = -72k^2b_4R^2 + 72k^2b_2 + 360k^2b_2\nu_r + 504b_4\nu_\theta R^2 - 216k^2b_4\nu_rR^2 - 504b_2\nu_\theta - 2520b_2 + 1512b_4R^2 - 3\rho h\omega^2R^2 \quad (70)$$

$$\Theta_6 = 84k^2b_3 - 3360b_3 - 672b_3\nu_\theta + 420k^2b_3\nu_r \quad (71)$$

$$\Theta_7 = 96k^2b_4 - 864b_4\nu_\theta + 480k^2b_4\nu_r + 2\rho h\omega^2 - 4320b_4 \quad (72)$$

From Eq. (72) we get the expression for natural frequency squared ω^2:

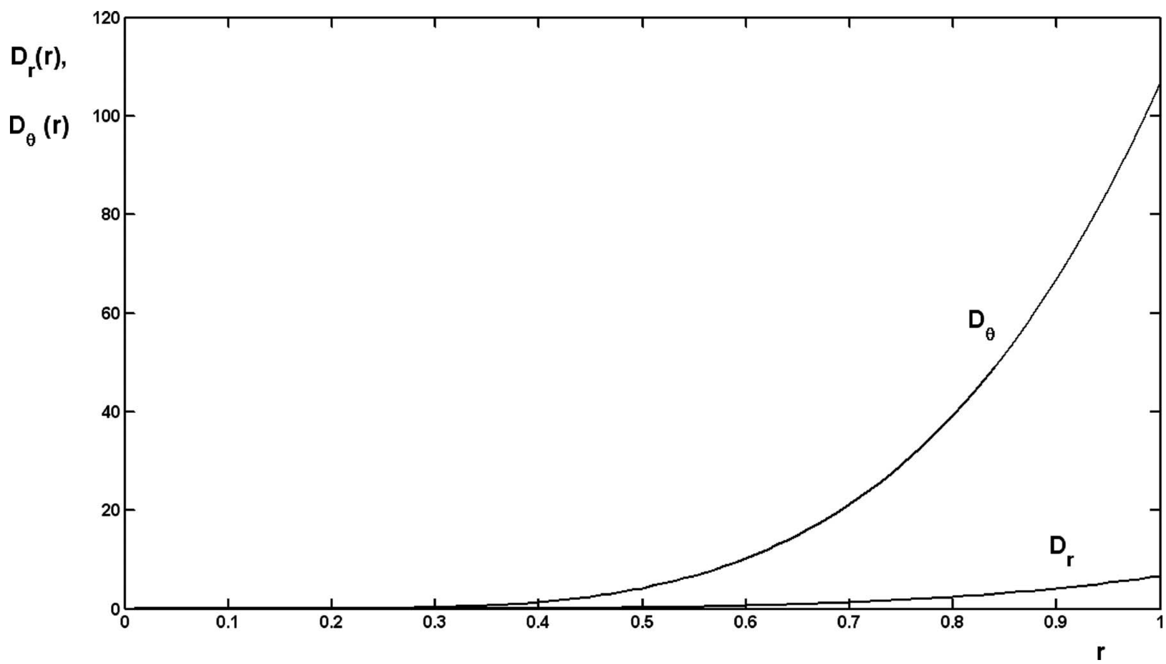

Fig. 10 Variation in $D_r(r)$ and $D_\theta(r)$ versus radial coordinate r for Eq. (83)

$$\omega^2 = -48(-45 - 9\nu_\theta + 5k^2\nu_r + k^2)b_4/\rho h \quad (73)$$

Eq. (71) shows that b_3 vanishes identically. Substitution of Eq. (73) into Eq. (70), results in the expression for b_2:

$$b_2 = -\frac{b_4R^2(k^2 - 11\nu_\theta + 7k^2\nu_r - 69)}{k^2 - 35 - 7\nu_\theta + 5k^2\nu_r} \quad (74)$$

Solution of Eq. (69) gives $b_1=0$. The equation resulting from substitution of Eq. (74) into Eq. (68) results in the expression for b_0:

$$b_0 = -\frac{b_4R^4(k^2 - 11\nu_\theta + 7k^2\nu_r - 69)(3k^2\nu_r + k^2 - 15 - 5\nu_\theta)}{(k^2 - 35 - 7\nu_\theta + 5k^2\nu_r)(k^2 - 25 + 5k^2\nu_r - 5\nu_\theta)} \quad (75)$$

Eq. (66) yields another expression for b_0:

$$b_0 = -2\frac{b_4R^4(k^2 - 9\nu_\theta + 5k^2\nu_r - 45)}{k^2 - 3\nu_\theta + 3k^2\nu_r - 9} \quad (76)$$

We demand the expressions for b_0 in Eqs. (75) and (76) to be equal. This results in the following quadratic equation for ν_r:

$$(8k^6 - 328k^4)\nu_r^2 + (-6k^6 + 508k^4 - 9718k^2)\nu_r - 117k^4 + 5359k^2 + k^6 - 69435 = 0 \quad (77)$$

For $k=1$ through $k=10$, Eq. (77) does not posses real positive roots for ν_r, except for $k=6$ for which it yields a positive root $\nu_r=0.9338$. Figure 8 shows the variation in $D(r)$ versus nondimensional radial r/R.

7 Example Section

Equations (15), (27), (47), and (73) all are in the analogous form:

$$\omega^2 = \varphi(\nu_r, \nu_\theta, k)b_4/\rho h \quad (78)$$

where the form of the function φ depends on the postulated mode shape or more specifically, for various values of m we have

$$\varphi(\nu_r, \nu_\theta, k) = \begin{cases} 24(21 + 7\nu_\theta - 3\nu_r k^2 - 21k^2), & \text{for } m = 1 \\ 24(21 + 7\nu_\theta - 3\nu_r k^2 - 21k^2), & \text{for } m = 2 \\ 35(32 + 8\nu_\theta - 4\nu_r k^2 - k^2), & \text{for } m = 4 \\ 48(45 + 9\nu_\theta - 5\nu_r k^2 + k^2), & \text{for } m = 5 \end{cases} \quad (79)$$

The coefficient b_4 was up to now treated as an arbitrary coefficient. We now demand that the circular plate's fundamental frequency must be equal prespecified value Ω, i.e.,

$$\omega = \Omega \quad (80)$$

This demand allows specification of the coefficient b_4:

$$b_4 = \Omega^2\rho h/\varphi(\nu_r, \nu_\theta, k) \quad (81)$$

Once coefficient b_4 is determined, the associated flexural rigidities $D_r(r)$ and $D_\theta(r)$ can be evaluated. Let, for example, $R=0.15$ m, $\rho=100$ kg/m^3, $h=0.08$ m, and the demanded natural frequency value $\Omega=95$ Hz. The tailoring can be accomplished by choosing $m=1$ ($k=2$, $\nu_r=0.35$), leading to $b_4=-52.40998839$. The appropriate flexural rigidities equal

$$D_r(r) = 52.41r^4 - 5.1886r^2 + 0.3279$$

$$D_\theta(r) = 209.64r^4 - 20.7544r^2 + 1.3116 \quad (82)$$

If, however, we choose $m=2$ ($k=4$, $\nu_r=0.1080$), we get $b_4=-9.764526153$. The corresponding flexural rigidities read

$$D_r(r) = 9.7645r^4 - 3.2921r^3 + 0.1641r^2 + 0.0492r$$

$$D_\theta(r) = 156.232r^4 - 52.6736r^3 + 2.6241r^2 + 0.7872r \quad (83)$$

As is seen, the vibration tailoring can be accomplished in several ways. Figures 9 and 10 show the variation in $D_r(r)$ and $D_\theta(r)$ as a function of r for Eqs. (82) and (83), respectively.

8 Conclusion

In this study vibration tailoring of a polar-orthotropic clamped plate has been accomplished via *analytical* calculations. Shirk et al. [1] emphasized that "…the structural dynamic behavior and flexibility of an aircraft are important to the performance and durability of a particular design, the development of these strategies assumes particular importance." It is hoped that the present method or its modifications will be applied also for other structural configurations.

Acknowledgment

I.E. acknowledges the partial financial support from J. M. Rubin Foundation at the Florida Atlantic University, and D.P. appreciates the partial financial support of Mr. Angelos Langadas Scholarship at the FAU.

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